Assume at \( t = 0 \) atom

has eigenstates \( \hat{H}_0 |k\rangle = E_k |k\rangle \)

\( t \geq 0 \) & field \& turned on

\( t > 0 \) and field turned on

\[ 1 = \langle \psi(0) | \psi(0) \rangle = \langle \psi(t) | \psi(t) \rangle = 1 = \sum_k |C_k(t)|^2 \]

\[ |C_k(t)|^2 \text{ is prob atom is in state } |k\rangle \text{ at time } t. \]

\[ \text{T.D.S.E.} \]

\[ \hat{H}|t\rangle = \sum_k C_k(t) e^{-i \frac{\hat{H}_0 t}{\hbar}} \]

\[ \hat{H} = \sum_k \left[ \hat{H}_0 + \hat{H}_x \right] = \sum_k \left[ \hat{H}_0 + \hat{H}_x \right] e^{-i \frac{\hat{H}_0 t}{\hbar}} \]

\[ \sum_k \left[ \hat{C}_k - i \frac{\hat{H}_0 \hat{C}_k}{\hbar} \right] e^{-i \frac{\hat{H}_0 t}{\hbar}} \]

\[ \Rightarrow \frac{i \hbar E_k}{\omega_k} \sum_k \langle k | \hat{H}_x | k \rangle \]

\[ \begin{align*}
\sum_k \langle k | \hat{H}_x | k \rangle & = C_k E_k + \sum_k \langle k | \hat{H}_x | k \rangle e^{i \omega_k t} \\
\Rightarrow \hat{C}_k & = -i \frac{E_k}{\omega_k} \sum_k \langle k | \hat{H}_x | k \rangle e^{i \omega_k t} \end{align*} \]

This gives us a first-order differential equation or, an entire infinite set of coupled differential equations. For realistic problem is to decide which subset of the states \( |k\rangle \) are important and truncate the sum at some large \( k' \) and then solve this numerically.
To get an analytic expression we use time-dependent (TOPT) perturbation theory. Typically we assume at $t=0$ that all the $C_{\ell}(t)\equiv 0$ except one $C_{\text{initial}}(t)$. That is, there is some state $\ell$ for which $C_{\text{initial}}(t) = C_{\ell}$ and $C_{\ell}(t) = 0$.

As time evolves the probe, to find the electron in some other state $C_{\text{final}}(t) \neq 0$ nonzero

$$P(\ell) = |C_{\ell}(t)|^2$$

where the relevant matrix element for the transition is

$$\langle \ell | \hat{H}_{\text{int}} | i \rangle = \langle \ell | (\hat{c}_i \cdot \vec{E}(t)) | i \rangle = \hat{H}^\dagger_{\ell i}$$

If we assume that $|\hat{H}^\dagger_{\ell i}|^2 \ll E_{\ell i} \equiv \hbar \omega_i - \hbar \omega_i$ that is the coupling of the atom to field is weak we can use TOPT. This is in Messiah.

$$0 \leq \lambda \leq 1$$

$$\Rightarrow C_{\ell}(t) = \sum_{m} \lambda^{m} C_{\ell}^{(m)}(t)$$

where this is a power series in $\lambda$ with coefficients $C_{\ell}^{(m)}(t)$.

With $C_{\ell}(t) = \sum_{m} \lambda^{m} C_{\ell}^{(m)}(t)$ we can substitute in 4.20 but also take

$$\langle \ell | \hat{H}_{\text{int}} | \ell \rangle = \hat{H}_{\ell \ell} \rightarrow \lambda \hat{H}_{\ell \ell}$$
\[
\frac{\partial}{\partial t} \left[ \sum_{m} \lambda^m C^{(m)}_k \right] = -i \gamma \sum_{\lambda} \lambda C^{(1)}_k \langle \lambda | H | \lambda \rangle e^{i \omega_\lambda t}
\]

\[
\Rightarrow \sum_{m} \lambda^m C^{(m)}_k = -i \gamma \sum_{\lambda} \lambda C^{(1)}_k H_{\lambda k} e^{i \omega_\lambda t} = -i \gamma \sum_{\lambda} \lambda C^{(1)}_k H_{\lambda k} e^{i \omega_\lambda t}
\]

We may equate powers of \( \lambda \) in \( \sum \lambda^m \).

\[
\lambda^0 \Rightarrow C^{(0)}_k = 0
\]

The \( \lambda^m \) implies no \( m \neq 0 \) term.

\[
\lambda' \Rightarrow C^{(1)}_k = -i \gamma \sum_{\lambda} C^{(0)}_k H_{\lambda k} e^{i \omega_\lambda t} \quad C^{(1)}_k \equiv C^{(1)}_k 
\]

\[
\lambda' \Rightarrow C^{(2)}_k = -i \gamma \sum_{\lambda} C^{(1)}_k H_{\lambda k} e^{i \omega_\lambda t}
\]

\[
\lambda' \Rightarrow C^{(n)}_k = -i \gamma \sum_{\lambda} C^{(n-1)}_k H_{\lambda k} e^{i \omega_\lambda t}
\]

So the idea is to solve the series by plugging in \( C^{(m)}_k \) to get \( C^{(m+1)}_k \). If we keep all \( m \) terms in series \( \sum_{m=0}^{\infty} \lambda^m C^{(m)}_k = C_k \) the solution is exact. Typically we truncate at \( m = 2 \) or \( 3 \).

Taking \( \lambda' \) we get to integrate and

\[
C^{(1)}_k(t) = -i \gamma \sum_{\lambda} \int_0^t dt' \sum_{\lambda} H_{\lambda k} e^{i \omega_\lambda t'}
\]

where \( \lambda^0 = \text{WLOG} \quad C^{(0)}_k(t) = 1 \quad \forall \lambda (OK) \)

Then if we take \( \ket{\lambda} \Rightarrow \langle \lambda | = 1 \text{initial} \rangle \); \( \langle \lambda | H | \lambda \rangle = H_{\lambda \lambda} \)

\( \Rightarrow \) if \( H_{\lambda \lambda} \) is small then at \( t = 0 \)

\( C^{(0)}_k(0) = 1 \) and \( C^{(1)}_k(0) = 0 \) for \( t > 0 \)

\[
|C^{(1)}_k(t)| \approx 1 \quad |C^{(1)}_k(t)| \ll 1
\]
Under this two level approximation we assume \( \omega_0 \gg \omega_{\text{fi}} \) and only terms \( |\nu_f = 1_i\rangle \) and \( |1_i > = |1\rangle \) contribute.

\[
\dot{C}_{1f} (t) = -i \gamma \int_0^t \text{d}t' H_{1f} (t') e^{i \omega_{1f} t} \tag{First order Eq.}
\]

Insert into \( \lambda \) to get

\[
\dot{C}_{1f} (t) = (-i \gamma)^2 \int_0^t \int_0^{t'} \text{d}t'' \sum_{\nu \nu'} H_{\nu \nu'} (t') e^{i \omega_{\nu \nu'} t''} e^{i \omega_{1f} t'}
\]

Recall that \( \langle f | \hat{\alpha} | i \rangle = e \int_0^t \text{d}t' \Psi_{n}^{*} = 0 \) if \( f = i \) (see Messiah). This connects states of opposite parity \( \Rightarrow H_{\nu \nu'} = 0 \) so that first order correction is

\[
\dot{C}_{1f} (t) = -i \gamma \int_0^t \text{d}t' H_{1f} (t') e^{i \omega_{1f} t'}
\]

Let \( H_{1} = -\vec{A} \cdot E_0 \omega_0 \text{ekt} \) be a field \( \propto \omega \).

We can integrate letting (semi-classical) \( \langle f | \hat{\alpha} | i \rangle = \frac{d}{dt} \Psi_{1f} \)

\[
\dot{C}_{1f} (t) = -i \gamma \left( E_0 \cdot \frac{d}{dt} \Psi_{1f} \right) \int_0^t e^{i \omega_{1f} t'} e^{-i (\omega - \omega_{\text{fi}}) t'}
\]

\[
= \frac{i \gamma}{2} (E_0 \cdot \frac{d}{dt} \Psi_{1f}) \left\{ \frac{e^{i \omega t - \frac{t}{2} \omega_0^2} - 1}{(\omega_0^2 - \omega^2)} + \frac{e^{-i \omega t - \frac{t}{2} \omega_0^2} - 1}{(\omega_0^2 - \omega^2)} \right\}
\]

If \( \omega_{\text{fi}} > 0 \) and \( \Delta = \omega - \omega_{\text{fi}} \ll \omega_0 \) the \( \omega_0 \) term is in resonance and the \( \omega_0 \) is out of resonance so we drop it. \( \Delta \) \( \propto \) a corotates with atom frequency so Rotating Wave Approx. (RWA).
Hence
\[
\frac{C_{T}(t)}{\xi_T} = -\frac{U_0^2}{\frac{\hbar}{2}} (\vec{E}_0 \cdot \vec{A}_T) e^{-\frac{\imath \Delta t}{2}} \left[ e^{\frac{\imath \Delta t}{2}} \frac{e^{-\frac{\imath \Delta t}{2}}}{e^{-\frac{\imath \Delta t}{2}} - e^{-\frac{\imath \Delta t}{2}}} \right]
\]
\[
= -\frac{U_0^2}{\frac{\hbar}{2}} (\vec{E}_0 \cdot \vec{A}_T) e^{-\frac{\imath \Delta t}{2}} \frac{e^{\imath \Delta t}}{e^{-\frac{\imath \Delta t}{2}}}
\]
\[
= \frac{2U_0^2}{\hbar} e^{-\frac{\imath \Delta t}{2}} \sin(\Delta t/2)
\]

\[\Downarrow\]
\[
P_{\xi_T}(t) \equiv \left| C_{\xi_T}(t) \right|^2 = \frac{y^2}{4} \frac{|\vec{E}_0 \cdot \vec{A}_T|^2}{(\Delta t/2)^2} \frac{\sin^2(\Delta t/2)}{(\Delta t/2)^2}
\]

To first order in \(\Delta\), this is only valid if the electron is in state \(|f\rangle\) at time \(t > 0\).

Let \(\text{sinc}(x) \equiv \frac{\sin x}{x}\).

Note \(\text{sinc}(0) = 1\) is maximum at \(x = 0\).

\[\text{L'Hopital's Rule:} \quad \frac{\sin x}{x} \to \frac{\cos x}{1} \to 1 \text{ as } x \to 0\]

\[
P_{\xi_T}(t) = \frac{1}{4} \frac{y^2}{U_0^2} \sin^2(\Delta t/2)
\]

\[
P_{\xi_T}(0) = 0
\]

Note: \(\Delta \to 0 \Rightarrow \)

\[
\left| P_{\xi_T}(t) \right|_{\text{max}} \xrightarrow{\Delta \to 0} \frac{y^2}{4} \frac{U_0^2}{\hbar} \frac{\Delta}{2}
\]

Now for \(\Delta \neq 0\)

\[
P_{\xi_T}(t) \propto \frac{\sin^2(\Delta t/2)}{\Delta^2}
\]

which is periodic with maxima when \(\Delta t/2 = \pi/2\) or \(t = \pi/\Delta\) atomic resonance beats or field \(\omega\) with beat period \(T = \pi/\Delta\) maximum at

\[
\left| P_{\xi_T}(t) \right|_{\text{max}} \left| \Delta \to 0 \right. \frac{1}{4} \frac{U_0^2}{\hbar} \left. \frac{\Delta}{2} \right|_{t = \pi n / \Delta}
\]
From this we can see maximum prob at $\Delta = 0$ for $\Delta \neq 0$ probability oscillates periodically from $|i\rangle \rightarrow |f\rangle \rightarrow |i\rangle$. These are Rabi oscillations. There are oscillations when $\Delta = 0$ but this theory can't handle it. We can see that as a func of $\Delta$ $P(t)$ is mostly peaked at $\Delta = 0$ with height $t^2$ and width $1/t$ so area $\approx (t^2)(1/t) \times t$.

More precisely

$$\sum_{\Delta} P(\Delta, t) = \int_{-\infty}^{\infty} \frac{1}{4\pi^2} U^2 \frac{\sin^2(\Delta t/\tau)}{(\Delta t/\tau)^2} \frac{t^2}{2} \left( \frac{\text{d} \Delta t}{\tau} \right)^2 \frac{1}{t}$$

$$= \int_{-\infty}^{\infty} \frac{2t}{4\pi^2} \frac{\sin^2 x}{x^2} = \frac{2t}{2\pi^2} \frac{\pi}{2} = \frac{\pi t U^2}{2\pi^2}$$

The area of the peak plus wiggles.

Using the sinc representation of Dirac $\delta$

$$\lim_{t \to \infty} \frac{\sin^2 (\Delta t/\tau)}{\Delta^2} \approx \frac{\tau}{2} t \delta (\Delta)$$

Hence for $t \gg 0$

$$P^{(1)}(t) \sim \frac{\pi}{2\pi^2} \frac{U^2}{t} \delta (\Delta)$$

$\Delta_k = \omega - \omega_{3i}$
The rate is defined as:

\[ W_{fi} = \frac{p_{fi}}{\tau} \]

which is independent of \( \tau \)

The rate is given by:

\[ W_{fi}^{(1)} = \frac{\pi}{2 \hbar^2} \frac{U_{fi}^2}{\hbar} \tilde{\Gamma}(\Delta_{fi}) \]

We did not assume only one transition \( |i\rangle \rightarrow |f\rangle \) participates. If a collection of final states \( |1f\rangle \) participates then

\[ W_{1f} = \frac{\pi}{2 \hbar^2} \sum_{ii} U_{ii}^2 \tilde{\Gamma}(\Delta_{fi}) \]

This is called Fermi's Golden Rule. Remember, we assumed only a single mode \( E \) for free space \( \tilde{\Gamma}(\Delta_{fi}) \) becomes \( p_{w_{fi}} \) the density of modes.

For thermal light, single mode, then

\[ U_{fi}^2 \rightarrow F(w) \equiv \left| \langle f|\hat{a}^\dagger \cdot \hat{E}_0(w) |i\rangle \right|^2 \]

since thermal light has random phase we can treat it as an independent set of drivers.

\[ W_{fi} \rightarrow p_{fi}^{(1)}(t) = \frac{1}{\hbar} \int_{-\infty}^{\infty} dw \, \frac{n}{\hbar^2} \left[ \frac{\Delta(w) t/c}{\Delta_{fi}^2(w)} \right] F(w) \]

since sine \( \sin \Delta(c\Delta) \) we can pull \( F(w) \) out of integral provided \( E(w) \) is broadband.

\[ W_{\delta_{fi}} = \frac{\pi}{2 \hbar^2} F(w_{fi}) = \frac{\pi}{2 \hbar^2} \left| \frac{1}{\hbar} \cdot \frac{\tilde{E}_0(w)}{\hbar} \cdot \frac{\Delta_{fi}^2(w)}{\Delta_{fi}^2} \right|^2 \]

**Thermal Light**